Some applications of system of coincidence theorems in Constrained equilibrium Problems
系統相容定理在控制平衡點問題之研究

林來居 、劉旭燿

摘 要

在本篇論文中，我們首先建立另外一種型態的系統固定點定理；而且利用 Ansari 和姚任之的系統固定點定理以及 Horvach 的 continuous selection 定理，我們建立了一些樊基型態的系統相容定理，並且利用這個結果去處理一些制控的平衡點定理以及制控的 Pareto 平衡點問題。

Contents

關鍵字：固定點、系統固定點、相容點、系統相容點、制控相量平衡點

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1. Introduction.

In 1966 [7], Fan established the well-known coincidence theorem and applied it to the well-known minimax principle. There are a lot of extensions and generalizations of this coincidence theorem, see [4], [8], [16]. Latter, Park, Ansari and Yao gave some coincidence theorem in the system situation, and applied them to quasi-variational inequality, equilibrium problems, section problems and so on.

Recently, much attention has been attracted to equilibrium probelms with vector payoffs in game theory; for instance, see [5], [6], [18], [19]. One of the reasons is that multicriteria models can be better applied to real-world situations.

In this paper, we first give another type of system of fixed point theorem, by using the system of fixed point theorem introduced by Ansari and Yao [2]. As application of the system of fixed point theorem of Ansari and Yao, and a continuous selection theroem of Horvach [9], we give several Ky Fan type system of coincidence theorems, and applied them to establish some contrained weight Nash type equilibrium theorem and constrained vector [resp. weak vector] Nash type equilibrium theorems. We consider the problem of finding $\bar{x} = (\bar{x}_i)_{i \in I} \in X$, $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$, $\bar{y}_i \in \bar{S}_i(\bar{x})$ and

$$z - \bar{z}_i \notin -intD_i \text{ for all } z_i \in F_i(\bar{x}, y_i) \text{ and for all } y_i \in \bar{S}_i(\bar{x})$$

where $\{X_i\}$ is a family of convex subsets each in topological vector spaces (in short t.v.s.) E_i, $\{Y_i\}$ is a family of convex subsets each in t.v.s. V_i, $\{Z_i\}$ is a family of t.v.s. and $D_i \subset Z_i$ is a closed convex solid cones. $X = \prod_{i \in I} X_i$,

$Y = \prod_{i \in I} Y_i$, $F_i : X \times Y_i \multimap Z_i$ is a payoff multimaps, $S_i : X \multimap Y_i$ and $T_i : Y \multimap X_i$ are constrained multimaps.

In this problem, we have two families (say family A and B) of players, and the payoff function is a vector valued multimap. For each strategy, we have many choice of vector-valued payoff from a multivalued payoff function F_i for the ith player in the family B. The payoff function F_i of the ith player in family B depends on the strategy combination of family A and the strategy of the ith player in family B.We want to find a strategy combination $\bar{x} \in \prod_{i \in I} X_i$ of family A and a strategy $\bar{y}_i \in Y_i$ of the ith player of family B, and choose a payoff $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ with relation $\bar{x}_i \in T_i(\bar{y})$ and $\bar{y}_i \in \bar{S}_i(\bar{x})$ such that payoff $\bar{z}_i$ of the ith player in family B is a weak minimum of $F_i(\bar{x}, \bar{S}_i(\bar{x}))$. This kind of problem is different from the Nash equilibrium problem and has some applications in real world.

In the Nash equilibrium problem, the payoff function is a single function, we cann't have many choice of payoff for each strategy. The payoff function depends on a strategy combination of one family of players, not on a strategy conbination of family A and the ith strategy of player B. In Nash equilibrium problem, the strategy does not subject to constrains. In our game the ith strategy of player B relates to the strategy combination of players in family A and the ith strategy of player in A realate to the strategy combination of players in family B.

In this paper, we apply the system coincidence theorem to establish the existence theorems of this kind of problem. We also establish some weighted equilibrium theorems.

2. Preliminaries.

A multimap $T : X \multimap Y$ is a map from a set X into the power set of a set Y. Let $T^{-} : Y \multimap X$ be defined by $x \in T^{-}y$ if and only if $y \in Tx$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be closed if its graph $G_r(T) = \{(x, y) | x \in X, y \in T(x)\}$ is closed in $X \times Y$; to have open fiber if $T^{-}(y)$ is open for all $y \in Y$; and to be compact if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y; to be upper semicontinuous (in short u.s.c.) if for every $x \in X$ and every open set V in Y with $T(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $T(u) \subset V$ for all $u \in U(x)$; to be lower semicontinuous (in short l.s.c.) if for every $x \in X$ and every neighborhood $V(y)$ of every $y \in T(x)$, there exists a neighborhood $U(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all $u \in U(x)$. and to be continuous if it is both u.s.c. and l.s.c..

Let Z a real topological vector space and D a convex solid cone in Z. let y_1, $y_2 \in Z$, we denote that

$$y_1 \leq y_2 \quad \text{if} \quad y_2 - y_1 \in D \quad \text{and}$$

$$y_i < y_2 \quad \text{if} \quad y_2 - y_1 \in intD.$$

Also we denote that $D^* = \{\varphi \in Z^* | \langle \varphi, u \rangle \geq 0 \text{ for all } u \in D\}$.

In this paper, for each given $m \in I\!N$, we denote by $I\!R^m_+$ the nonnegative orthant of $I\!R^m$, i.e.,

$$I\!R^m_+ := \{u = (u^1, ...u^m) \in I\!R^m \text{ such that } u^j \geq 0 \text{ for } j = 1, 2....m\}.$$

and,

$$intI\!R^m_+ := \{u = (u^1, ...u^m) \in I\!R^m : u^j > 0 \text{ for all } j = 1, 2....m\}.$$

Let I be an finite or infinite index set. For each $i \in I$, let $X = \prod_{i\in I} X_i$, $X^i = \prod_{j\in I, j\neq i} X_j$ and we write $X = X^i \times X_i$. For each $x \in X$, $x_i \in X_i$ denote the ith coordinate and $x^i \in X^i$ the projection of X on X^i and we also write $x = (x^i, x_i)$.

The following definitions and lemmas are needed in this paper.

Definition 1[3]. Let Z a real topological vector space and D a convex solid cone in Z. A function $f : Z \to I\!R$ is said to be monotonically increasing (resp. strictly monotonically increasing) with respect to D if $f(a) \geq f(b)$ for all $a - b \in D$ (resp. $f(a) > f(b)$ for all $a - b \in intD$)

Definition 2[15]. Let X be a nonempty subset of a Hausdorff topological vector space E, Y be a nonempty subset of a topological vector space V. We say $F \in DKT(X, Y)$ if Y is convex, and there exists a multimap $B : X \multimap Y$ with $\mathrm{co}(B(x)) \subset F(x)$ for all $x \in X$, $B(x) \neq \emptyset$ for all $x \in X$ and the fibers $B^{-}(y)$ are open (in X) for each $y \in Y$.

Definition 3[12,14]. Let X be a nonempty convex subset of a real topological vector space E, Z a real topological vector space and D a convex solid cone in Z. Let $F : X \multimap Z$ be a multimap, F is said to be D-convex if for every x_1, $x_2 \in X$, $\lambda \in [0, 1]$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ there exists $y_3 \in F(\lambda x_1 + (1 - \lambda)x_2)$ such that

$$\lambda y_1 + (1 - \lambda)y_2 \geq_D y_3.$$

F is said to be D-quasiconvex if for any $z \in Z$, the set

$$\{x \in X| \text{ there is a } y \in F(x) \text{ such that } z - y \in D\} \text{ is convex.}$$

Definition 4. A strategy combination $(\bar{x}, \bar{y}) \in X \times Y$ is call a weak contrained vector equilibrium of a game $(N, \{X_i\}_{i\in \mathbb{N}}, \{Y_i\}_{i\in \mathbb{N}}, \{F_i\}_{i\in \mathbb{N}}, \{S_i\}_{i\in \mathbb{N}}, \{T_i\}_{i\in \mathbb{N}})$ if $\bar{x} = (\bar{x}_i)_{i\in N} \in \prod_{i\in \mathbb{N}} X_i = X$, $\bar{y} = (\bar{y}_i)_{i\in \mathbb{N}} \in \prod_{i\in \mathbb{N}} Y_i$ such that $\bar{x}_i \in T_i(\bar{y})$, $\bar{y}_i \in S_i(\bar{x})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying $z_i - \bar{z}_i \notin -int\mathbb{R}^{k_i}$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in \bar{S}_i(\bar{x})$, where $F_i : X \times Y_i \multimap \mathbb{R}^{k_i}$, $S_i : X \multimap Y_i$ and $T_i : Y \multimap X_i$ are multimaps for each $i \in \mathbb{N}$, and $\bar{S}_i : X \multimap Y_i$ is defined by $\bar{S}_i(x) = \overline{S}_i(x)$ for all $x \in X$.

Definition 5. A strategy combination $(\bar{x}, \bar{y}) \in X \times Y$ is call a contrained weight equilibrium with respect to the weight vector $W = (W_1, ...W_n)$ of a game $G = (X_i, Y_i, F_i, S_i, T_i)_{i\in \mathbb{N}}$ if $\bar{x} = (\bar{x}_i)_{i\in \mathbb{N}} \in \prod_{i\in \mathbb{N}} X_i = X$, $\bar{y} = (\bar{y}_i)_{i\in \mathbb{N}} \in \prod_{i\in \mathbb{N}} Y_i$ such that $\bar{x}_i \in T_i(\bar{y})$, $\bar{y}_i \in \bar{S}_i(\bar{x})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

(i) $W_i \in \mathbb{R}^{k_i}_+ \setminus \{0\}$; and

(ii) $W_i \cdot (\bar{z}_i) \leq W_i \cdot (z_i)$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in \bar{S}_i(\bar{x})$, where F_i, S_i, T_i and $\bar{S}_i$ are same as Definition 4, and " $\cdot$ " denotes the inner product.

Remark. (a) In particular, when $W_i \in T^{k_i}_+$ for all $i \in \mathbb{N}$, the strategy $\bar{x}$ is said to be a normalized constrained weight equilibrium with respect to W, where $T^{k_i}_+$ is a simplex of $\mathbb{R}^{k_i}$, i.e.,

$$T^{k_i}_+ := \{u = (u^1, ...u^n) \in \mathbb{R}^{k_i}_+ | \sum_{j=1}^{k_i} u^j = 1\}.$$

Lemma 1[2]. Let I be an index set and for each $i \in I$, let $\{E_i\}$ be a family of Hausdorff t.v.s.. $\{X_i\}$ be a family of nonempty convex subsets with each in E_i. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let H_i, $F_i : X \multimap X_i$ be two multimaps. Suppose that for each $i \in I$

(i) for each $x \in X$, $\text{co}F_i(x) \subset H_i(x)$;

(ii) $X = \bigcup\{intF_i^-(x_i) : x_i \in X_i\}$

(iii) if X is not compact, assume that there exist a nonempty compact convex subset $C_i \in X_i$ and a nonempty compact subset B of X such that for each $x \in X \setminus B$ there exists $y_i \in C_i$ such that $x \in int_X S_i^-(y_i)$

Then there exists a $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i \in H_i(\bar{x})$ for each $i \in I$.

Remark. If X is compact, then (iii)' is true automatically.

Lemma 2[9]. Let X be a compact space, Y a convex subset of a topological vector space, H, $F : X \multimap Y$. Suppose that

(i) for each $x \in X$, $M \in \langle F(x) \rangle$ implies $\text{co}M \subset H(x)$;

(ii) $X = \bigcup\{intF^-(y) : y \in Y\}$

Then there is a continuous function $f : X \to Y$ such that $f(x) \in H(x)$ for all $x \in X$.

Lemma 3[13]. Let Z be a topological vector space, D a closed convex solid cone. Then for any fixed $e \in intD$ and $z \in Z$,

$$p(y) = \min\{t \in I\!R | y \in z + te - D\}$$

is a continuous and strictly monotonically increasing function from Z to $I\!R$.

Lemma 4[1]. Let X and Y be two t.v.s., $T : X \multimap Y$ be a multimap.

(a) If T is an u.s.c. multimap with closed values, then T is closed.

(b) If X is compact and T is u.s.c. with compact values, then $T(X)$ is compact.

Lemma 5[10]. Let X be a convex subset of topological vector space E, Z be a topological vector space, D a closed convex solid cone. Let $F : X \multimap Z$ be a multimap. For any fixed $e \in intD$ and $z \in Z$, let

$$p(y) = \min\{t \in I\!R | y \in z + te - D\}$$

If F is D-quasiconvex, then $pF : X \multimap I\!R$ is $I\!R^+$-quasiconvex.

3. System of fixed points and coincidence theorems.

As a simple consequences of system of fixed point of Ansari and Yao [2], we obtain another type of system of fixed point theorem.

Theorem 3.1. Let $\{X_i\}$ be a family of nonempty convex subsets each in a locally convex quasi-complete t.v.s. E_i. For each $i \in I$, let F_i, H_i: $X \multimap X_i$ be multimaps such that H_i is compact. Suppose that for each $i \in I$

(i) for each $x \in X$, $\mathrm{co}F_i(x) \subset H_i(x)$;

(ii) $X = \bigcup\{intF_i^-(x_i) : x_i \in X_i\}$

Then there exists a $\bar{x} \in X$ such that $\bar{x} \in H(\bar{x}) = \prod_{i \in I} H_i(\bar{x})$; that is, $\bar{x}_i \in H_i(\bar{x})$ for each $i \in I$.

Proof. Since for each $i \in I$, H_i is compact, there exists a compact subset $Q_i \subset X_i$ such that $H_i(X) \subset Q_i$ for each $i \in I$. Let K_i=coQ_i then $K = \prod_{i \in I} K_i \subset X$ is compact and convex.
By (i), for each $i \in I$ and for each $x \in K$, $\mathrm{co}F_i(x) \subset H_i(x)$. For each $i \in I$, $F_i(x) \subset H_i(x) \subset Q_i \subset K_i$ for all $x \in K$, and $F_i^-(y_i) = \emptyset$ for all $y_i \in X_i \setminus K_i$. By (ii), for each $i \in I$, $K = \bigcup\{int_K F_i^-(x_i) : x_i \in X_i\} = \bigcup\{int_K F_i^-(x_i) : x_i \in K_i\}$. Since K is compact and convex, then by Lemma 1, there exists a $\bar{x} \in K$ such that $\bar{x} \in H(\bar{x})$.Then the proof is complete.

As a simple consequence of Theorem 3.1, we have the following Corollary.

Corollary 3.1[15]. Let $\{X_i\}$ be a family of nonempty convex subsets each in a locally convex quasi-complete t.v.s. E_i. For each $i \in I$, let $H_i \in DKT(X, X_i)$ is a compact multimap.Then H has a fixed point.

Proof. Since for each $i \in I$ $H_i \in DKT(X, X_i)$, then by the definition of $DKT(X, X_i)$ there exists a multimap $F_i : X \multimap X_i$ such that co$F_i(x) \subset H_i(x)$ for all $x \in X$, $F_i(x) \neq \emptyset$ for all $x \in X$ and the fibers $F_i^{-}(x_i)$ are open in X for all $x_i \in X_i$. Since for each $i \in I$ and for all $x \in X$, $F_i(x) \neq \emptyset$ and $F_i^{-}(x_i)$ is open for all $x_i \in X_i$, for each $i \in I$, $X = \bigcup\{intF_i^{-}(x_i) : x_i \in X_i\}$. Therefore all conditions of theorem 1 are satisfied, hence there exists a $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x_i} \in H_i(\bar{x})$ for each $i \in I$.

As a consequence of Lemma 1 and a continuous selection theorem of Horvach [9], we have the following coincidence theorems.

Theorem 3.2. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty convex subsets each in quasi-complete t.v.s. E_i and V_i respectively. For each $i \in I$, let F_i, $H_i : X \multimap Y_i$; S_i, $T_i : Y \multimap X_i$ be multimaps . Suppose that for each $i \in I$,

(i) for each $x \in X$, co$F_i(x) \subset H_i(x)$;

(ii) $X = \bigcup\{intF_i^{-}(y_i) : y_i \in Y_i\}$;

(iii) for each $y \in Y$, co$S_i(y) \subset T_i(y)$;

(iv) $Y = \bigcup\{intS_i^{-}(x_i) : x_i \in X_i\}$

(v) T_i is compact

Then there eaist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$, for each $i \in I$.

Proof. Since for each $i \in I$, T_i is compact, there exists a compact subset $Q_i \subset X_i$ such that $T_i(Y) \subset Q_i$ for each $i \in I$. Let $K_i =$coQ_i and $K = \prod_{i\in I} K_i$, then K is a compact and convex subset in X.
By (i), for each $x \in K$, co$F_i(x) \subset H_i(x)$;
By (ii), $K = \bigcup\{int_K F_i^{-}(y_i) : y_i \in Y_i\}$. It follows from Lemma 2 that for each $i \in I$, there exists a continuous function $f_i : K \to Y_i$ such that $f_i(x) \in H_i(x)$ for all $x \in K$. Let $f : K \to Y$ be defined by $f(x) = \prod_{i\in I} f_i(x)$, and let $P_i,W_i : K \multimap K_i$ be defined by $W_i(x) = S_i(f(x))$ and $P_i(x) = T_i(f(x))$ for all $x \in K$.It is easy to see $W_i^{-} = f^{-1}(S_i^{-}(x_i))$.
By assumption (iii) for each $i \in I$ and for all $x \in X$, co$W_i(x) =$co$S_i(f(x)) \subset T_i(f(x)) = P_i(x)$ and since $S_i(Y) \subset T_i(Y) \subset Q_i \subset K_i$, hence by assumption (iv) and the continuity of f

$$\begin{aligned} K = f^{-1}(Y) &= f^{-1}[\bigcup\{intS_i^{-}(x_i) : x_i \in X_i\}] \\ &= f^{-1}[\bigcup\{intS_i^{-}(x_i) : x_i \in K_i\}] \\ &\subset \bigcup\{intf^{-1}(S_i^{-}(x_i)) : x_i \in K_i\} \\ &= \bigcup\{intW_i^{-}(x_i) : x_i \in K_i\} \subset K. \end{aligned}$$

Hence,

$$K = \bigcup\{intW_i^{-}(x_i) : x_i \in K_i\}.$$

Then by Lemma 1, there exists a $\bar{x}_i \in T_i(f(\bar{x}))$. Let $\bar{y} = (\bar{y}_i)_{i\in I}$ such that $\bar{y} = f(\bar{x})$, then for each $i \in I$, $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$. Therefore the proof is complete.

Theorem 3.3. Let $\{X_i\}$ be family of nonempty convex subsets each in t.v.s. E_i. For each $i \in I$, let F_i, $H_i : X \multimap X_i$; S_i, $T_i : X \multimap X_i$ be multimaps . Suppose that for each $i \in I$,

(i) for each $x \in X$, $\text{co}F_i(x) \subset H_i(x)$;

(ii) $X = \bigcup\{intF_i^-(y_i) : y_i \in X_i\}$;

(iii) there exists a nonempty compact subset K of X and compact convex subset C_i of X_i such that for each $x \in X \setminus K$, there exists a $\tilde{x}_i \in C_i$ such that $x \in intF_i^-(\tilde{x}_i)$;

(iv) for each $y \in X$, $\text{co}S_i(y) \subset T_i(y)$;

(v) $X = \bigcup\{intS_i^-(x_i) : x_i \in C_i\}$

Then there eaist $\bar{x} = (\bar{x})_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that for each $i \in I$, $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$.

Proof. Since K is compact and $K \subset X \subset \bigcup\{intF_i^-(y_i) : y_i \in X_i\}$. then there exist $\{y_i^1, y_i^2 ... y_i^n\} \in X_i$ such that

$$K \subset \bigcup_{j=1}^{n}\{intF_i^-(y_i^j)\}$$

From (iii) we have

$$X \setminus K \subset \bigcup\{intF_i^-(x_i) : x_i \in C_i\}$$

For each $i \in I$, let $Q_i = \text{co}\{C_i \cup \{y_i^1, y_i^2 ... y_i^n\}\}$, since C_i is compact convex, then Q_i is compact and convex for each $i \in I$. Let $Q = \prod_{i \in I} Q_i$, then Q is also compact and convex. Hence, for each $i \in I$

$$Q \setminus K \subset \bigcup\{intF_i^-(x_i) : x_i \in C_i\} \subset \bigcup\{intF_i^-(x_i) : x_i \in Q_i\}$$

and,

$$K \subset \bigcup_{j=1}^{n}\{intF_i^-(y_i^j)\} \subset \bigcup\{intF_i^-(x_i) : x_i \in Q_i\}$$

therefore,

$$Q \subset \bigcup\{intF_i^-(x_i) : x_i \in Q_i\}$$

so

$$Q = \bigcup\{int_Q F_i^-(x_i) : x_i \in Q_i\}$$

Since Q is compact and convex, it follows from Lemma 2 that for each $i \in I$, there exists a continuous function $f_i : Q \to Q_i$ such that $f_i(x) \in H_i(x)$ for all $x \in Q$. Let $f : Q \to Q$ be defined by $f(x) = \prod_{i \in I} f_i(x)$, and let $P_i, W_i : Q \multimap Q_i$ be defined by $W_i(x) = S_i(f(x))$ and $P_i(x) = T_i(f(x))$ for all $x \in Q$. It is easy to see $W_i^- = f^{-1}(S_i^-(x_i))$.

By (v)

$$X = \bigcup\{intS_i^-(y) : y \in C_i\} \subset \bigcup\{intS_i^-(y) : y \in Q_i\}$$

hence,

$$Q = \bigcup\{int_Q S_i^-(y) : y \in Q_i\}$$

By assumption (iv) for each $i \in I$ and for all $x \in X$, $\text{co}W_i(x) = \text{co}S_i(f(x)) \subset T_i(f(x)) = P_i(x)$, hence by assumption and the continuity of f

$$Q = f^{-1}(Q) = f^{-1}[\bigcup\{int_Q S_i^-(y) : y \in Q_i\}]$$

$$\subset \bigcup\{int_Q f^{-1}(S_i{}^-(y)) : y \in Q_i\}$$

$$= \bigcup\{int_Q W_i{}^-(y) : y \in Q_i\} \subset Q$$

Hence,

$$Q = \bigcup\{int_Q W_i{}^-(y) : y \in Q_i\}.$$

Then by Lemma 1, there exists a $\bar{x}_i \in P_i(\bar{x}) = T_i(f(\bar{x}))$. Let $\bar{y} = (\bar{y}_i)_{i\in I}$ such that $\bar{y} = f(\bar{x})$, then for each $i \in I$, $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$.

Theorem 3.4. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty convex subsets each in t.v.s. E_i and V_i respectively. For each $i \in I$, let F_i, $H_i : X \multimap Y_i$; S_i, $T_i : Y \multimap X_i$ be multimaps. Suppose that for each $i \in I$,

(i) for each $x \in X$, $\mathrm{co}\prod_{i\in I} F_i(x) \subset \prod_{i\in I} H_i(x)$;

(ii) $X = \bigcup\{int(\prod_{i\in I} F_i)^-(y) : y \in Y\}$;

(iii) there exists a nonempty compact subset B os X and a nonempty compact convex subset K_i of X_i such that for each $x \in X \setminus B$, there exists a $\tilde{y}_i \in K_i$ such that $x \in int_X(S_i \circ \prod_{i\in I} F_i)^-(\tilde{y}_i)$;

(iv) for each $y \in Y$, $\mathrm{co}S_i(y) \subset T_i(y)$; and

(v) $Y = \bigcup\{intS_i{}^-(x_i) : x_i \in X_i\}$

Then there eaist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$, for each $i \in I$

Proof. Define two multimaps $\tilde{S}_i$, $\tilde{T}_i : X \multimap X_i$ by

$$\tilde{S}_i(x) = S_i \circ \prod_{i\in I} F_i(x) \quad \text{and} \quad \tilde{T}_i(x) = T_i \circ \prod_{i\in I} H_i(x)$$

Let $x \in X$, by (ii), for each $i \in I$ there exists a $y \in Y$ such that $x \in int(\prod_{i\in I} F_i)^{-}(y)$. Since $y \in Y$, by (v) there exists a $x_i' \in X_i$ such that $y = (y_i)_{i\in I} \in intS_i^{-}(x_i')$. Hence

$$x \in int(\prod_{i\in I} F_i)^{-}(y) \subset int(\prod_{i\in I} F_i)^{-}(intS_i^{-}(x_i')) \subset int(\prod_{i\in I} F_i)^{-}(S_i^{-}(x_i'))$$

$$= int(S_i \circ \prod_{i\in I} F_i)^{-}(x_i') = int\tilde{S}_i^{-}(x_i')$$

So

$$X \subset \bigcup\{int\tilde{S}_i^{-}(x_i') : x_i' \in X_i\} \subset X$$

Hence we have

$$X = \bigcup\{int\tilde{S}_i^{-}(x_i) : x_i \in X_i\}$$

Now we wnat to show that for each $i \in I$ and for each $x \in X$, $\text{co}\tilde{S}_i(x) \subset \tilde{T}_i(x)$. Indeed, for each $i \in I$ if $x_i \notin \tilde{T}_i(x) = T_i \circ \prod_{i\in I} H_i(x)$, then $x_i \notin T_i(y)$ for all $y \in \prod_{i\in I} H_i(x)$. By condition (iv), $x_i \notin \text{co}(S_i(y))$ for all $y \in \prod_{i\in I} H_i(x)$, hence $x_i \notin \text{co}(S_i(\prod_{i\in I} H_i(x)))$.

From (i), $x_i \notin \text{co}(S_i(\prod_{i\in I} F_i(x))) = \text{co}\tilde{S}_i(x)$. Therefore for each $i \in I$ and for each $x \in X$, $\text{co}\tilde{S}_i(x) \subset \tilde{T}_i(x)$. By (iii) for each $x \in X \setminus B$ there exists a $\tilde{y}_i \in K_i$ such that $x \in int_X(\tilde{S})^{-}(\tilde{y}_i)$.

Hence by Lemma 1, there exists a $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ such that for each $i \in I$, $\bar{x}_i \in \tilde{T}_i(\bar{x}) = T_i \circ \prod_{i\in I} H_i(\bar{x})$.

Therefore there exists a $\bar{y} = (\bar{y}_i)_{i\in I}$ such that for each $i \in I$, $\bar{y} \in \prod_{i\in I} H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$. Therefore for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and $\bar{y}_i \in H_i(\bar{x})$.

As a simple consequence of Theorem 3.2, we have the following Corollary.

Corollary 3.2. Let $\{X_i\}$ be a family of nonempty compact convex subsets each in t.v.s. E_i. $\{Y_i\}$ be a family of nonempty convex subsets each in t.v.s. V_i. For each $i \in I$, let F_i, $H_i : X \multimap Y_i$; S_i, $T_i : Y \multimap X_i$ be multimaps. Suppose that for each $i \in I$,

(i) for each $x \in X$, $\mathrm{co}F_i(x) \subset H_i(x)$;

(ii) $X = \bigcup\{intF_i^-(y_i) : y_i \in Y_i\}$;

(iii) for each $y \in Y$, $\mathrm{co}S_i(y) \subset T_i(y)$;

(iv) $Y = \bigcup\{intS_i^-(x_i) : x_i \in X_i\}$

Then there eaist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in H_i(\bar{x})$ and $\bar{x}_i \in T_i(\bar{y})$, for each $i \in I$.

Proof. Since for each $i \in I$, X_i is compact, $X = \prod_{i\in I} X_i$ is a compact set. So condition (v) of Theorem 3.2 is automatically true. Hence the conclusion follows from Theorem 3.2.

As a simple consequence of Corollary 3.2, we have the following Corollary.

Corollary 3.3. Let $\{X_i\}$ be a family of nonempty compact convex subsets each in t.v.s. E_i. $\{Y_i\}$ be a family of nonempty convex subsets each in t.v.s. V_i. For each $i \in I$, let $F_i : X \multimap Y_i$; $S_i : Y \multimap X_i$ be multimaps. Suppose that for each $i \in I$,

(i) for each $x \in X$, $F_i(x)$ is a nonempty convex set;

(ii) for each $y_i \in Y_i$, $F_i^-(y_i)$ is open;

(iii) for each $y \in Y$, $S_i(y)$is a nonempty convex cet; and

(iv) for each $x_i \in X_i$, $S_i^-(x_i)$ is open

Then there eaist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in F_i(\bar{x})$ and $\bar{x}_i \in S_i(\bar{y})$, for each $i \in I$.

Remark. If I is singleton, then Corollary 3.3 reduce to Ky Fan coincidence theorem [7].

4.Weak constrained Nashed type vector equilibrium theorems.

As applications of system of Coincidence theorem, we establish weak constrained Nash type vector equilibrium theorems.

Theorem 4.1. Let $\{X_i\}$ be a family of nonempty compact convex subsets each in normed space E_i. $\{Y_i\}$ be a family of nonempty compact convex subsets each in normed space V_i. $\{Z_i\}$ be a family of real normed space. For each $i \in I$, $D_i \subset Z_i$ be close convex solid cone and $D_i \neq Z_i$. Suppose that for each $i \in I$

(i) $F_i : X \times Y_i \multimap Z_i$ is a continuous multimap with compact values;

(ii) for each fixed $x \in X$, $y_i \multimap F_i(x, y_i)$ is D_i-quasiconvex;

(iii) $S_i : X \multimap Y_i$ with open fibers and for all $x \in X$, $S_i(x)$ is a nonempty convex set;

(iv) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^-(x_i) : x_i \in X_i\}$;

(v) $\bar{S}_i$ is u.s.c. where $\bar{S}_i(x) = \overline{S_i(x)}$;

(vi) $\{P_i\}$ is family of continuous monotone strictly increasing function from $Z_i \to I\!R$, as defined in Lemma 3.

Then there exist $\bar{x} \in X$, $\bar{y} \in Y$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and $\bar{y}_i \in \bar{S}_i(\bar{x})$ satisfying

(1) $P_i(\bar{z}_i) \leq P_i(z_i)$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in \bar{S}_i(\bar{x})$;

(2) $z_i - \bar{z}_i \notin -intD_i$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in \bar{S}_i(\bar{x})$.

Proof. Define $H_{i,n} : X \multimap Y_i$ by

$$H_{i,n}(x) = \{y_i \in S_i(x) | minP_iF_i(x, y_i) < minP_iF_i(x, \bar{S}_i(x)) + \frac{1}{n}\}$$

Following the same argument as Theorem 8 [12], we show that for each $i \in I$ and for each $x \in X$ $H_{i,n}(x)$ is a nonempty set. It follows from (ii) and (iv), $y_i| - \circ P_iF_i(x, y_i)$ is $I\!R^+$-quasiconvex, $H_{i,n}(x)$ is convex for each $i \in I$ and for each $x \in X$, and $H^-_{i,n}(y_i)$ is open for each $i \in I$ and for each $y_i \in Y_i$. Therefore $X = \bigcup\{intH^-_{i,n}(y_i) : y_i \in Y_i\}$. Hence by Corollary 2, there exist $\bar{x}^{(n)} = (\bar{x}_{i,n})_{i\in I}$, $\bar{y}^{(n)} = (\bar{y}_{i,n})_{i\in I}$ such that $\bar{x}_{i,n} \in T_i(\bar{y}^{(n)})$, $\bar{y}_{i,n} \in S_i(\bar{x}^{(n)})$, and

$$minP_iF_i(\bar{x}^{(n)}, \bar{y}_{i,n}) < minP_iF_i(\bar{x}^{(n)}, \bar{S}_i(\bar{x}^{(n)}) + \frac{1}{n}, \text{ for each } i \in I$$

As in Theorem 8 [12], let $\bar{z}_{i,n} \in F_i(\bar{x}^{(n)}, \bar{y}_{i,n})$ satisfying

$$P_i(\bar{z}_{i,n}) = minP_iF_i(\bar{x}^{(n)}, \bar{y}_{i,n})$$

Since for each $i \in I$ $\bar{S}_i$ is compact, and for each $i \in I$ F_i is u.s.c. with compact values, it follows lemma 4 that for each $i \in I$, $F_i(X, \bar{S}_i(X))$ is compact. Since $\bar{x}^{(n)} = (\bar{x}_{i,n})_{i\in I} \in X$, $\bar{y}^{(n)} = (\bar{y}_{i,n})_{i\in I} \in Y$ and $\bar{z}_{i,n} \in F_i(\bar{x}^{(n)}, \bar{S}_i(\bar{x}^{(n)}))$, there exist subsequence $\{\bar{x}^{n(\alpha)}\}$ of $\{\bar{x}^{(n)}\}$, $\{\bar{y}_{i,n(\alpha)}\}$ of $\{\bar{y}_{i,n}\}$ and $\{\bar{z}_{i,n(\alpha)}\}$ of $\{\bar{z}_{i,n}\}$, $\bar{x} \in X$ $\bar{y}_i \in Y_i$ and $\bar{z}_i \in F_i(X, \bar{S}_i(X))$ such that $\bar{x}^{n(\alpha)} \to \bar{x}$, $\bar{y}_{i,n(\alpha)} \to \bar{y}_i$ and $\bar{z}_{i,n(\alpha)} \to \bar{z}_i$ For each $\alpha \in \Gamma$ and for each $i \in I$

$$P_i(\bar{z}_{i,n(\alpha)}) < minP_iF_i(\bar{x}^{n(\alpha)}, \bar{S}_i(\bar{x}^{n(\alpha)})) + \frac{1}{n(\alpha)}.$$

Letting $n(\alpha) \to \infty$,

then by the continuity of the function $x \to minP_iF_i(x, \bar{S}_i(x))$ for each $i \in I$, we have

$$P_i(\bar{z}_i) \le minP_iF_i(\bar{x}, \bar{S}_i(\bar{x})) \qquad \text{for each } i \in I.$$

That is,

$$P_i(\bar{z}_i) = minP_iF_i(\bar{x}, \bar{S}_i(\bar{x})) \qquad \text{for each } i \in I.$$

Since T_i is closed, $\bar{x}_{i,n} \in T_i(\bar{y}^{(n)})$, so $\bar{x}_i \in T_i(\bar{y})$. By (v), $\bar{S}_i$ is u.s.c. with closed values, it follows from Lemma 4 that $\bar{S}_i$ is closed for each $i \in I$, hence $\bar{y}_i \in \bar{S}_i(\bar{x})$ for each $i \in I$. Thus for each $i \in I$, there exist $\bar{y}_i \in Y_i$, $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such taht $\bar{x}_i \in T_i(\bar{y})$, $\bar{y}_i \in \bar{S}_i(\bar{x})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

$$P_i(\bar{z}_i) \le P_i(z_i) \text{ for all } z_i \in F_i(\bar{x}, y_i) \text{ and for all } y_i \in \bar{S}_i(\bar{x})$$

Since for each $i \in I$, P_i is monotone strictly increasing, then

$$z_i - \bar{z}_i \notin -intD_i \text{ for all } z_i \in F_i(\bar{x}, y_i) \text{ and for all } y_i \in \bar{S}_i(\bar{x})$$

Remark. In Theorem 4.1, we don't assume that $S_i : X \multimap Y_i$ is an u.s.c. with closed values and $S_i^-(y_i)$ is open for all $y_i \in Y_i$. It was pointed by Lin and Park [11], if we assume this, then $S_i(x) = Y_i$ for all $x \in X$.

In Theorem 4.1, if F is single value function, then we have Corollary 4.1.

Corollary 4.1. Let $\{X_i\}$ be a family of nonempty compact convex subsets each in normed space E_i. $\{Y_i\}$ be a family of nonempty compact convex subsets each in normed space V_i. $\{Z_i\}$ be a family of real normed space. For each $i \in I$, $D_i \subset Z_i$ be close convex solid cone and $D_i \neq Z_i$. Suppose that for each $i \in I$

(i) $F_i : X \times Y_i \to Z_i$ is a continuous multimap with compact values;

(ii) for each fixed $x \in X$, $y_i| \to F_i(x, y_i)$ is D_i-quasiconvex;

(iii) $S_i : X \multimap Y_i$ with open fibers and for all $x \in X$, $S_i(x)$ is a nonempty convex set;

(iv) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^{-1}(x_i) : x_i \in X_i\}$;

(v) $\bar{S}_i$ is u.s.c. where $\bar{S}_i(x) = \overline{S_i(x)}$;

(vi) $\{P_i\}$ is family of continuous monotone strictly increasing function from $Z_i \to \mathbb{R}$, as defined in Lemma 3.

Then there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and $\bar{y}_i \in \bar{S}_i(\bar{x})$; and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

(1) $P_i \circ F_i(\bar{x}, \bar{y}_i) \leq P_i \circ F_i(\bar{x}, y_i)$ for all $y_i \in Y_i$;

(2) $F_i(\bar{x}, y_i) - F_i(\bar{x}, \bar{y}_i) \notin -intD_i$ for all $y_i \in Y_i$.

Now we establish some constrained equilibrium theorms, and apply it to the constrained Pareto equilibrium and Pareto equilibrium.

Theorem 4.2. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty compact convex subsets each in normed space E_i and V_i respectively.$\{Z_i\}$ be a family of real normed space . For each $i \in I$, $D_i \subset Z_i$ be close convex solid cone and $D_i \neq Z_i$. Suppose that for each $i \in I$

(i) $F_i : X \times Y_i \multimap Z_i$ is continuous with compact values;

(ii) for each fixed $x \in X$, $y_i \multimap F_i(x, y_i)$ is D_i-convex;

(iii) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^-(x_i) : x_i \in X_i\}$;

(iv) $P_i \in D_i^* \setminus \{0\}$.

Then there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that for each $i \in I$, $\bar{x_i} \in T_i(\bar{y})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

(1) $P_i(\bar{z}_i) \leq P_i(z_i)$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in Y_i$;

(2) $z_i - \bar{z}_i \notin -intD_i$ for all $z_i \in F_i(\bar{x}, y_i)$ and for all $y_i \in Y_i$.

Proof. For each $i \in I$, define $S_i : X \multimap Y_i$ by $S_i(x) = Y_i$ for each $x \in X$, then for each $i \in I$, $S_i : X \multimap Y_i$ is a multimap with open fibers and for all $x \in X$, $S_i(x)$ is a convex set. Also $\bar{S}_i$ is u.s.c., then following the same method of Theorem 4.1, we have the conclusion.

Corollary 4.2. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty compact convex subsets each in normed space E_i and V_i respectively.$\{Z_i\}$ be a family of real normed space . For each $i \in I$ $D_i \subset Z_i$ be close convex solid cone and $D_i \neq Z_i$. Suppose that for each $i \in I$

(i) $F_i : X \times Y_i \to Z_i$ is continuous function;

(ii) for each fixed $x \in X$, $y_i \to F_i(x, y_i)$ is D_i-convex;

(iii) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^-(x_i) : x_i \in X_i\}$;

(iv) $P_i \in D_i^* \setminus \{0\}$

Then there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y}_i)$ satisfying

(1) $P_i \circ F_i(\bar{x}, \bar{y}_i) \leq P_i \circ F_i(\bar{x}, y_i)$ for all $y_i \in Y_i$;

(2) $F_i(\bar{x}, y_i) - F_i(\bar{x}, \bar{y}_i) \notin -intD_i$ for all $y_i \in Y_i$.

Proof. The conclusion follows immediately from Theorem 4.2 if we let $\hat{F}_i(x, y_i) = \{F_i(x, y_i)\}$

As application of Theorem 4.2, we have the following vector valued weight Nash equilibrium theorem.

Theorem 4.3. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty compact convex subsets each in normed space E_i and V_i respectively. For each $i \in I$, let $F_i : X \times Y_i \multimap I\!R^{k_i}$ be payoff function, and there is a weight combination $W = (W_i)_{i \in I}$ with $W_i \in I\!R^{k_i}_{+} \setminus \{0\}$. Suppose that for each $i \in I$

(i) F_i is continuous with compact values;

(ii) for each fixed $x \in X$, $y_i \multimap F_i(x, y_i)$ is $I\!R^{k_i}_{+}$-convex;

(iii) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^{-}(x_i) : x_i \in X_i\}$;

Then there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

$$\langle W_i, \bar{z}_i \rangle \le \langle W_i, z_i \rangle \text{ for all } z_i \in F_i(\bar{x}, y_i), \text{ and for all } y_i \in Y_i$$

Proof. For each $i \in I$, let $P_i(z_i) = \langle W_i, z_i \rangle$ for all $z_i \in I\!R^{k_i}$. Then $P_i \in D_i^* \setminus \{0\}$ for each $i \in I$. Therefore if we take $Z_i = I\!R^{k_i}$ and $D_i = I\!R^{k_i}_{+}$, then the conditions of Theorem 4.2 are satisfied. Hence we have for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and $\bar{z}_i \in F_i(\bar{x}, \bar{y}_i)$ satisfying

$$\langle W_i, \bar{z}_i \rangle \le \langle W_i, z_i \rangle \text{ for all } z_i \in F_i(\bar{x}, y_i), \text{ and for all } y_i \in Y_i$$

As a simple consequence of Theorem 4.3, we have the constrained weight equilibrium

Theorem 4.4. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty compact convex subsets each in normed space E_i and V_i respectively. For each $i \in I$, let $F_i : X \times Y_i \to I\!R^{k_i}$ be payoff function, and there is a weight combination $W = (W_i)_{i \in I}$ with $W_i \in I\!R^{k_i}_+ \setminus \{0\}$. Suppose that for each $i \in I$

(i) F_i is a continuous function;

(ii) for each fixed $x \in X$, $y_i \to F_i(x, y_i)$ is $I\!R^{k_i}_+$-convex;

(iii) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^-(x_i) : x_i \in X_i\}$;

Then there exists at least one constrained weight equilibrium with the weight combination $W = (W_i)_{i \in I}$.

Proof. If we take $\hat{F}_i(x, y_i) = \{F_i(x, y_i)\}$, then the conditions of Theorem 4.3 are all satisfied. Hence by Theorem 4.3, there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and

$$W_i \circ F_i(\bar{x}, \bar{y}_i) \leq W_i \circ F_i(\bar{x}, y_i) \qquad \text{for all } y_i \in Y_i$$

Therefore $(\bar{x}, \bar{y}) \in X \times Y$ is a constrained weight equilibrium with the weight combination $W = (W_i)_{i \in I}$.

As the same argument of Theorem 4.4, we have the following constrained Pareto equilibrium theorem.

Theorem 4.5. Let $\{X_i\}$ and $\{Y_i\}$ be families of nonempty compact convex subsets each in normed space E_i and V_i respectively. For each $i \in I$, let $F_i : X \times Y_i \to I\!R^{k_i}$ be payoff function, and there is a weight combination $W = (W_i)_{i \in I}$ with $W_i \in T_+^{k_i}$. Suppose that for each $i \in I$

(i) F_i is continuous;

(ii) for each fixed $x \in X$, $y_i \to F_i(x, y_i)$ is $I\!R_+^{k_i}$-convex;

(iii) $T_i : Y \multimap X_i$ is a close multimap such that for all $y \in Y$, $T_i(y)$ is a nonempty convex set and $Y = \bigcup\{intT_i^-(x_i) : x_i \in X_i\}$;

Then for each $i \in I$, $\bar{x}_i \in T_i(\bar{y})$ and there exists at least one weak constrained Pareto equilibrium.Furthermore, if $W_i \in intT_+^{k_i}$ for each $i \in I$, then there exists at least one constrained Pareto equilibrium.

Proof. The proof follows immediately from Lemma 2.1 [18] and Theorem 4.4.

Remark. In Theorem 4.4 and 4.5, if we let $X = X^i$, $\{Y_i\} = \{X_i\}$ and $T_i : X^i \to X_i$ by $T_i(x^i) = X_i$ for all $x^i \in X^i$, then Theorem 4.4 and 4.5 can be reduce to the Pareto [resp. weak Pareto] equilibrium theorem with non-constrained cases. Furthermore, if the payoff function is real single-valued, then our problem can be reduced to the Nash equilibrium problem.

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