Exponential stability criterion for a class of nonlinear discrete systems

Ruey-Shyan Gau[a,1] Yeong-Jeu Sun[b] Jer-Guang Hsieh[a]

[a]Department of Electrical Engineering, National Sun Yat-Sen University
Kaohsiung, Taiwan 804, R.O.C.
[b]Department of Electrical Engineering, I-Shou University,
Kaohsiung County, Taiwan 840, R.O.C.

Abstract--In this paper, a criterion is proposed to guarantee the exponential stability for a class of nonlinear discrete systems. A numerical example is provided to illustrate the main result.

1. Introduction

During the last twenty years, digital computers have increased dramatically the applications of digital signal processing. In the meanwhile, the stability analysis of discrete systems has also been an active area of research; see, for example, [1]-[3], and the references therein. This is due to theoretical interests as well as to a powerful tool for practical system analysis and control design, since discrete systems appear in various engineering systems, such as image acquisition systems, digital speech transmission systems, seismic sounding systems, digital control systems, and radar signal processing. In the past, there have been a number of interesting developments in searching the stability criteria for discrete systems, but most were restricted to criteria for uncertain linear systems or criteria for nonlinear systems without any uncertainties; see, for example, [1]-[3]. It is the purpose of this paper to derive a criterion guaranteeing the exponential stability for a class of nonlinear discrete systems. For

[1] Corresponding author. E-mail: gau@mail.ksvs.kh.edu.tw; g8431807@mail.ee.nsysu.edu.tw

convenience, in this letter, Z^+ denotes the set of all non-negative integers, $\underline{k}$ denotes the set $\{1, 2, \cdots, k\}$, $\bar{k}$ denotes the set $\{0, 1, 2, \cdots, k\}$, and $\|A\|$ denotes the induced Euclidean norm of the matrix A.

2. Main result

Consider the following nonlinear discrete system:

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \tag{1a}$$

$$x(0) = x_0, \tag{1b}$$

where $x \in \Re^n$ and $f(\cdot, 0) = 0$.

Definition: The system (1) is said to be (globally) exponentially stable if there exist positive numbers r, $0 < r < 1$, and K such that, for all $x(0) \in \Re^n$,

$$\|x(n)\| \le K \cdot \|x(0)\| \cdot r^n, \quad \forall n \in Z^+.$$

In this case, the positive number r is called the exponential decay rate.

Before presenting our main result, we make an assumption as follows.

(A1) There exist nonnegative functions $V(n)$ and $r_i(n) \ge 0$, $\forall i \in \underline{k}$, $n \in Z^+$, and nonnegative constants λ_1, λ_2, p, c_i, with $0 < \lambda_1 \le \lambda_2$, $p > 0$, $c_i \ge 1$, $\forall i \in \underline{k}$, such that

$$\lambda_1 \|x(n)\|^p \le V(n) \le \lambda_2 \|x(n)\|^p, \quad n \in Z^+, \tag{2}$$

and the function V along the solutions of (1) satisfies

$$V(n+1) \le \sum_{i=1}^{k} r_i(n) \cdot V^{c_i}(n), \quad \forall n \in Z^+. \tag{3}$$

Now, we present the main result for the exponential stability of (1) as follows.

Theorem 1: The system (1) with (A1) is exponentially stable provided that there exist constants $J \in Z^+$ and $q \ge 0$ such that

$$\sum_{i=1}^{k} r_i(n+J) \cdot V^{c_i - 1}(J) \le q < 1, \quad \forall n \in Z^+. \tag{4}$$

In this case, the guaranteed exponential decay rate is given by $q^{1/p}$.

Proof: By (3), one has

$$V(n+1) \le \left[\sum_{i=1}^{k} r_i(n) \cdot V^{c_i-1}(n)\right] \cdot V(n), \quad \forall n \ge J. \tag{5}$$

From (5) and setting $n = J$, it can be deduced that

$$V(J+1) \le q \cdot V(J),$$

in view of (4). Similarly, we may show that

$$V(m+1) \le q \cdot V(m), \quad \forall m \ge J. \tag{6}$$

This shows that $V(m)$ is s strictly decreasing sequence after $m \ge J$. By (5) and setting $n = J+1$, it can be deduced that

$$\begin{aligned} V(J+2) &\le \left[\sum_{i=1}^{k} r_i(J+1) \cdot V^{c_i-1}(J+1)\right] \cdot V(J+1) \\ &\le \left[\sum_{i=1}^{k} r_i(J+1) \cdot V^{c_i-1}(J)\right] \cdot V(J+1) \\ &\le q \cdot V(J+1) \\ &\le q^2 \cdot V(J), \end{aligned} \tag{7}$$

in view of (4) and (6). Similarly, by (5) and setting $n = J+2$, it can be deduced that

$$\begin{aligned} V(J+3) &\le \left[\sum_{i=1}^{k} r_i(J+2) \cdot V^{c_i-1}(J+2)\right] \cdot V(J+2) \\ &\le \left[\sum_{i=1}^{k} r_i(J+2) \cdot V^{c_i-1}(J)\right] \cdot V(J+2) \\ &\le q \cdot V(J+2) \\ &\le q^3 \cdot V(J), \end{aligned} \tag{8}$$

in view of (4) and (7). By similar reasoning as in (7) and (8), we have

$$V(J+n) \le q^n \cdot V(J), \quad \forall n \in Z^+. \tag{9}$$

By (9) and (2), we have

$$\|x(J+n)\| \le q^{n/p} (\lambda_2/\lambda_1)^{1/p} \|x(J)\|, \quad \forall n \in Z^+. \tag{10}$$

Note that if $x(0) = 0$, then $x(k) = 0$ for all $k \in Z^+$. In this case, we set $K_1 = 0$. Now assume $x(0) \ne 0$. Set

$$K_1 := \max\left[1, \max_{j=1}^{J}\left(q^{-j/p} \|x(0)\|^{-1} \|x(j)\|\right)\right].$$

Then we have

$$\|x(k)\| \le K_1 \cdot (\lambda_2/\lambda_1)^{1/p} \cdot \|x(0)\| \cdot \left[q^{1/p}\right]^k, \quad \forall k \in \bar{J}. \tag{11}$$

From (10) and (11), we have

$$\|x(k)\| \le K_1 \cdot (\lambda_2/\lambda_1)^{2/p} \cdot \|x(0)\| \cdot \left[q^{1/p}\right]^k, \quad \forall k \in Z^+.$$

This completes our proof.

3. Illustrative example

Consider the following uncertain nonlinear discrete system

$$\begin{aligned} x(n+1) &= \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} \\ &= \begin{bmatrix} [\Delta a \cos(x_2(n))]x_2^2(n) + \Delta b x_1^2(n) x_2^2(n) + [\Delta c \cos(x_2(n))]x_1^4(n) \\ [\Delta d \sin(x_1(n))]x_2^4(n) + \Delta e x_1^2(n) x_2^2(n) + [\Delta f \sin(x_2(n))]x_1^2(n) \end{bmatrix}, \quad \forall n \in Z^+, \end{aligned} \tag{12a}$$

$$x(0) = x_0, \tag{12b}$$

where

$$-0.4 \le \Delta a,\ \Delta f \le 0.4, \quad -0.1 \le \Delta b,\ \Delta c,\ \Delta d,\ \Delta e \le 0.1.$$

Selecting

$$V(n) = \|x(n)\|, \quad p = 1, \quad \lambda_1 = \lambda_2 = 1, \quad k = 2,$$

$$r_1(n) = 0.4, \quad r_2(n) = 0.1, \quad c_1 = 2, \quad c_2 = 4,$$

then (A1) is satisfied. With the choice $J = 0$, one has

$$\sum_{i=1}^{2} r_i(n) \cdot V^{c_i - 1}(0) \le 0.6 + 0.3375, \quad \forall n \in Z^+,$$

which implies that (4) is satisfied. Consequently, by Theorem 1 with $q = 0.9375$, we conclude that the system (12) is exponentially stable with the guaranteed exponential decay rate $r = 0.9375$. For simulation purpose, we set $x_1(0) = 1.2$, $x_2(0) = 0.6$, $\Delta a = 0.4$, $\Delta f = -0.4$, $\Delta b = -0.1$, $\Delta c = -0.1$, $\Delta d = 0.1$, and $\Delta e = 0.1$, the state responses are shown in Fig.1 and Fig. 2.

Remark 1: It is noted that the asymptotic stability of (12) cannot be guaranteed by the main results of [3]. Furthermore, the exponential stability of (12) cannot be guaranteed by the main results of [2].

4. Conclusion

In this paper, a criterion has been provided to guarantee the exponential stability for a class of nonlinear discrete systems. A numerical example has also been given to illustrate the main result. It is interesting to consider the criteria for the exponential stability of more general uncertain nonlinear discrete systems.

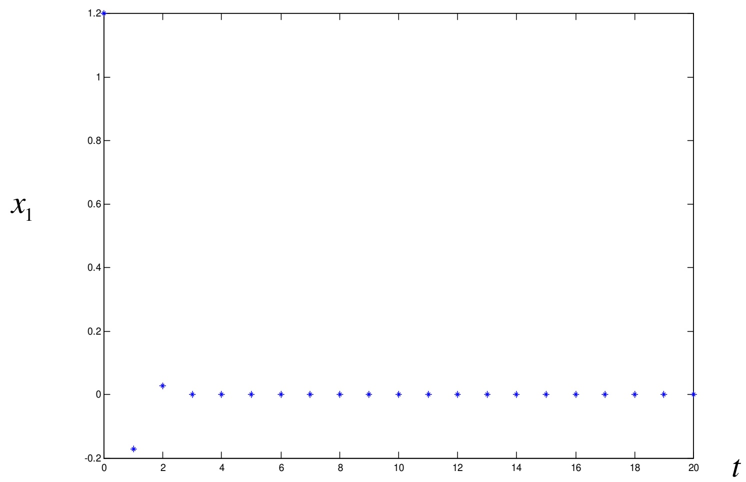

Fig 1. The trajectory of state x_1 for system (12).

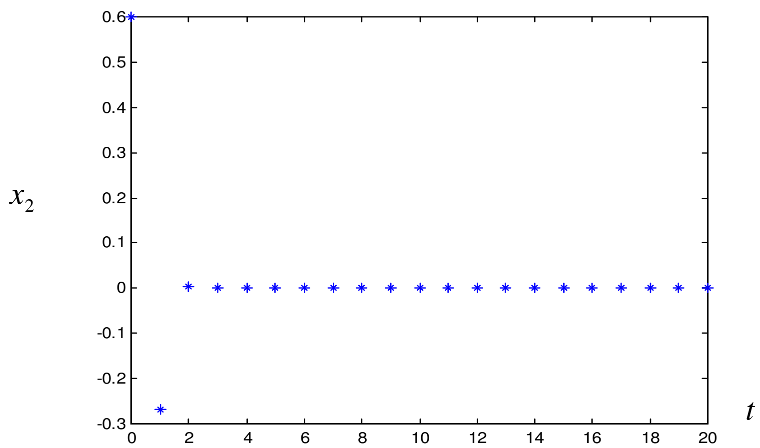

Fig 2. The trajectory of state x_2 for system (12).

References

1 Kucera, V.: *Analysis and Design of Discrete Linear Control System* (Prentice-Hall, NJ, 1991) pp. 66-81

2 Sun, Y.J., and Yu, G.J.: "Delay-dependent exponential stability criteria for nonlinear time-varying discrete systems with multiple time delays," *Journal of the Franklin Institute, Engineering and Applied Mathematics*, 1997, **334**, (4), pp. 659-666

3 Tsao, T.C.: "Simple stability criteria for nonlinear time-varying discrete systems," *Systems & Control Letters*, 1994, **22**, (1), pp. 223-225