# 維持系統穩定下的最大延遲時間上限之研究

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## 摘要

對於標準二階無延遲控制系統，欲維持漸近穩定下，我們推導系統允許延遲的最大上限，最後，我們以一實例來應証並說明主要結果的貢獻。

關鍵詞：漸近穩(Asymptotic stability)，延遲系統(delay system)，最大上限制(upper bound)

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# Admissible upper bound for arbitrary constant delay without destroying stability

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## Abstract

**For a given standard second order control systems, which is free of delay, an upper bound of arbitrary constant delay introduced to the control system is given such that the asymptotic stability is preserved. A numerical example is also given to illustrate the main result.**

## Introduction and main result:

For a stable standard second order control systems, it is usually true that the asymptotic stability will not be destroyed if some sufficiently small delays are introduced to such system [1]. For a given stable standard second order control systems with delayed output feedback, it is interesting to know an upper bound for the delay, under which the asymptotic stability is still guaranteed.

Specifically, we consider, in this letter, the following standard second order control systems:

$$\frac{L[y(t)]}{L[u(t)]} = \frac{Y(s)}{U(s)} = G_o(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2},$$

where $U(s)$ is the Laplace transform of the system input $u(t)$, $Y(s)$ is the Laplace transform of the system output $y(t)$, $\xi$ is the damping ratio, and $w_n$ is the natural frequency, with $\xi > 0$ and $w_n > 0$.

Remark 1: It is noted that, the system (1) is asymptotically stable in view of $\xi > 0$ and $w_n > 0$.

As we know, the performance, e.g., resonant peak and bandwidth, of system (1) can be improved

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by unity output feedback. In practical control system, the unity output feedback are usually replaced by the unity delayed output feedback because the energy in the system propagate with a finite speed. Thus, the purpose of this letter is to propose an upper bound of the delay $\tau$ introduced to the control system (1) with unity delayed output feedback, depicted in Fig. 1, such that the asymptotic stability is preserved.

Now we present our main result as follows.

Theorem 1: The system (1) with unity delayed output feedback is asymptotically stable, depicted in Fig. 1, provided that $0 \le \tau < \tau^*$, where

$$\tau^* := \begin{cases} \frac{1}{w_n\sqrt{2-4\zeta^2}}\sin^{-1}\left[2\zeta\sqrt{2-4\zeta^2}\right], & if\, 0<\zeta<\frac{\sqrt{2}}{2}; \\ \infty, & otherwise. \end{cases} \tag{1}$$

Proof: The characteristic equation of the closed-loop systems, depicted in Fig. 1, is given by

$$f(s) := s^2 + 2\zeta w_n s + e^{-\tau s} w_n^2 + w_n^2 = 0 \tag{2}$$

Firstly, all the roots of $f(s)$ lie in the negative half-plane under the case of $\tau = 0$. Secondly, the roots of $f(s)$ are either real or occur in complex conjugate pairs. As $\tau$ varies, the roots will move and the solution of (2) is a continuous function of $\tau$, with $\tau \ge 0$. Thus the goal of this proof is that of determining for what values of $\tau$, if any, there are roots actually on the imaginary axis. Setting $f(jw^*) = 0$ and equating real and imaginary parts leads to the pair of equation

$$\begin{cases} -(w^*)^2 + w_n^2\cos(\tau w^*) + w_n^2 = 0; \\ 2\zeta w^* w_n - w_n^2 \sin(\tau w^*) = 0. \end{cases} \tag{3}$$

Case 1: $\left(0<\zeta<\frac{\sqrt{2}}{2}\right)$

It can be deduced, from (3), that

$$(w^*,\tau) = \left(\pm w_n\sqrt{2-4\zeta^2}, \frac{1}{w_n\sqrt{2-4\zeta^2}}\sin^{-1}\left[2\zeta\sqrt{2-4\zeta^2}\right]\right),$$

which implies that there are zeros of $f(s)$ lying on the imaginary axis in the case of $\tau = \tau^*$, and all zeros of $f(s)$ lie in the left half-plane in the case of $\tau \in [0, \tau^*)$.

Case 2: $\left(\zeta \ge \frac{\sqrt{2}}{2}\right)$

There is no solution of $(w^*, \tau)$ satisfying (3) in the case of $\zeta \geq \frac{\sqrt{2}}{2}$ . Consequently, the closed-loop system is asymptotically stable for $0 \leq \tau < \infty$ . This completes our proof.

*Illustrative example:* Consider the system (1) with . $G_o(s) = \frac{1}{s^2 + s + 1}$ Clearly, one has $\zeta$ =0.5 and $w_n$ =1 .By Theorem 1, we conclude that the system (1) with the unity delayed output feedback $u(t) = r(t)$ - y( $t$- $\tau$ ) , depicted as Fig. 1, is asymptotically stable provided that $0 \leq \tau < \tau^* = \frac{\pi}{2}$ .

*Conclusion:* In this letter, an upper bound of time delay has been provided such that the asymptotic stability of the standard second order control systems with unity delayed output feedback is preserved. A numerical example has also been provided to illustrate the use of our main result. It is of interest to estimate an upper bound of the delay term for more general systems with unity delayed output feedback.

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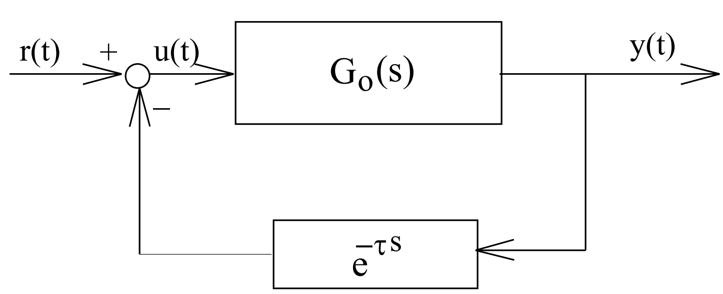


Fig. 1 The feedback-controlled system.